

NOTES ON PERIODIC ELEMENTS OF GARSIDE GROUPS

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ABSTRACT. Let G be a Garside group with Garside element Δ . An element g in G is said to be *periodic* if some power of g lies in the cyclic group generated by Δ . This paper shows the following. (i) The periodicity of an element does not depend on the choice of a particular Garside structure if and only if the center of G is cyclic. (ii) If $g^k = \Delta^{ka}$ for some nonzero integer k , then g is conjugate to Δ^a . (iii) Every finite subgroup of the quotient group $G/\langle\Delta^m\rangle$ is cyclic, where Δ^m is the minimal positive central power of Δ .

Keywords: Garside group; periodic element; braid group.

2000 Mathematics Subject Classification: Primary 20F36; Secondary 20F10

1. INTRODUCTION

The class of Garside groups, first introduced by Dehornoy and Paris [DP99], is a lattice-theoretic generalization of braid groups and Artin groups of finite type. Recently, there were several results on periodic elements of Garside groups such as the characterization of finite subgroups of the central quotient of finite type Artin groups by Bestvina [Bes99] and its extension to Garside groups by Charney, Meier and Whittlesey [CMW04]; a new algorithm for solving the conjugacy search problem for periodic braids by Birman, Gebhardt and González-Meneses [BGG07]; the characterization of periodic elements in the braid groups of complex reflection groups by Bessis [Bes06a].

This paper is interested in some questions on periodic elements in Garside groups. Let G be a Garside group with Garside element Δ . An element g in G is said to be *periodic* (with respect to Δ) if either g is the identity or it is commensurable with Δ , that is, g^k and Δ^ℓ are conjugate for some nonzero integers k and ℓ [Bes06b, BGG08].

1.1. Periodicity and Garside structure. The periodicity of an element in a Garside group generally depends on the choice of a particular Garside structure, more precisely on the Garside element. A Garside group may admit more than one Garside structure. For example, the n -strand braid group B_n has two standard Garside structures: the classical and dual Garside structures [Thu92, BKL98]. Therefore it is natural to ask the following question.

When is the periodicity of an element independent of the choice of a particular Garside structure?

It is easy to see that the periodicity does not depend on the choice of a Garside structure if and only if any two Garside elements (arising from different Garside structures) are commensurable, and that this happens if the center $Z(G)$ is cyclic. We show that the converse is also true.

Theorem 3.1. *Let G be a Garside group. Then $Z(G)$ is cyclic if and only if any pair of Garside elements of G are commensurable.*

The irreducible Artin groups of finite type and, more generally, the braid groups of irreducible well-generated complex reflection groups are Garside groups with cyclic center [Bes06a]. Therefore, in these groups, an element is periodic (with respect to a Garside element) if and only if it has a central power. However, not all Garside groups have cyclic center. A typical example is \mathbb{Z}^ℓ for $\ell \geq 2$.

1.2. p/q -periodicity, uniqueness of roots and a question of Bessis. First, we introduce a definition of Bessis in [Bes06b]: an element $g \in G$ is p/q -periodic if $g^q = \Delta^p$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$.

Note that $g^k = \Delta^\ell$ for some $k \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}$ if and only if g^q is conjugate to Δ^p for some $q \in \mathbb{Z}_{\geq 1}$ and $p \in \mathbb{Z}$, because Δ^m is central for some $m \geq 1$. Using this equivalence, we define the notion of p/q -periodicity in a slightly different way.

Definition 1.1. An element $g \in G$ is said to be p/q -periodic for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$ if g^q is conjugate to Δ^p and q is the smallest positive integer such that g^q is conjugate to a power of Δ .

In the above definition, the p/q -periodicity *a priori* depends on the actual p and q and not just on the rational number p/q because it may happen that g^{kq} is conjugate to Δ^{kp} for some $k \geq 2$ but g^q is not conjugate to Δ^p . Motivated by this observation, we show the following.

Theorem 4.3. Let G be a Garside group with Garside element Δ , and let $g \in G$ and $a, b, k \in \mathbb{Z}_{\neq 0}$.

- (i) If g^{kb} is conjugate to Δ^{ka} , then g^b is conjugate to Δ^a .
- (ii) If each of g^a and g^b is conjugate to a power of Δ , then so is $g^{\gcd(a,b)}$.

By this theorem, the term ‘ p/q -periodic’ contains that p and q are coprime.

The above theorem is a sort of uniqueness property of roots up to conjugacy. On this property, stronger results are known for some specific groups. Let g and h be elements of a group G such that

$$(1) \quad g^k = h^k \quad \text{for some } k \neq 0.$$

If G is the pure n -braid group P_n , then $g = h$ by Bardakov [Bar92]. (This also follows from the biorderability of the pure braid groups by Kim and Rolfsen [KR03].) If G is the n -braid group B_n , then g and h are conjugate by González-Meneses [Gon03]. If G is the Artin group of type \mathbf{B} , $\tilde{\mathbf{A}}$ or $\tilde{\mathbf{C}}$, then g and h are conjugate [LL10]. If G is the braid group of a well-generated complex reflection group and g and h are periodic elements, then g and h are conjugate by Bessis [Bes06a]. For a study of roots in mapping class groups, see [BP09].

Theorem 4.3 shows that if G is a Garside group and h is a power of Δ , then (1) implies that g and h are conjugate. In Garside groups, even for periodic elements, it is hard to obtain a result stronger than Theorem 4.3. For every $k \geq 2$, there is a Garside group with periodic elements g and h such that $g^k = h^k$ but g and h are not conjugate. (See Example 4.4.)

The following is a question of Bessis [Bes06b, Question 4].

Question. Let G be a Garside group with Garside element Δ . Let $g \in G$ be a periodic element with respect to Δ . Does G admit a Garside structure with Garside element g ?

The above question is answered almost positively in the case of the braid group B_n : each periodic element in B_n is conjugate to a power of one of the particular braids δ and ε which are the Garside elements in the dual Garside structures of B_n and $A(\mathbf{B}_{n-1})$, respectively, where $A(\mathbf{B}_{n-1})$ denotes the Artin group of type \mathbf{B}_{n-1} viewed as a subgroup of B_n . In [Bes06b], Bessis showed that the above question is answered almost positively in Garside groupoid setting.

To a Garside group G with an affirmative answer to the above question, the idea of Birman, Gebhardt and González-Meneses in [BGG07] can possibly be applied. Precisely, in order to solve the conjugacy search problem for periodic elements g and h of G , it suffices to find a Garside structure with Garside element g .

Using Theorem 4.3, we give a negative answer to the above question: there is a Garside group G with a periodic element g such that there is no Garside structure on G with Garside element g . (See Example 4.4.)

1.3. Finite subgroups of the quotient group G_Δ . Let m be the smallest positive integer such that Δ^m is central in G . Let G_Δ be the quotient $G/\langle\Delta^m\rangle$, where $\langle\Delta^m\rangle$ is the cyclic group generated by Δ^m . For an element $g \in G$, let \bar{g} denote the image of g under the natural projection from G to G_Δ . Hence, an element $g \in G$ is periodic if and only if \bar{g} has a finite order in G_Δ .

About finite subgroups of G_Δ , the following facts are known.

- (i) If G is an Artin group of finite type, then every finite subgroup of G_Δ is cyclic.
- (ii) If G is a Garside group, then every finite subgroup of G_Δ is abelian of rank at most 2.

The first was proved by Bestvina [Bes99, Theorem 4.5] and the second by Charney, Meier and Whittlesey [CMW04, Corollary 6.8] following the arguments of Bestvina. For the full statement of their results, see §5.

We show that Bestvina's result holds also for all Garside groups.

Theorem 5.3. *Let G be a Garside group with Garside element Δ . Then every finite subgroup of G_Δ is cyclic.*

Our proof uses the result of Charney, Meier and Whittlesey. Actually we prove that every finite abelian subgroup of G_Δ is cyclic. Because every finite subgroup of G_Δ is abelian, this implies the above theorem.

2. REVIEW OF GARSIDE GROUPS

This section describes basic definitions and properties of Garside groups. See [DP99, Deh02] for details.

For a monoid M , let 1 denote the identity element. An element $a \in M \setminus \{1\}$ is called an *atom* if $a = bc$ for $b, c \in M$ implies either $b = 1$ or $c = 1$. For $a \in M$, let $\|a\|$ be the supremum of the lengths of all expressions of a in terms of atoms. The monoid M is said to be *atomic* if it is generated by its atoms and $\|a\| < \infty$ for any element a of M . In an atomic monoid M , there are partial orders \leq_L and \leq_R : $a \leq_L b$ if $ac = b$ for some $c \in M$; $a \leq_R b$ if $ca = b$ for some $c \in M$.

Definition 2.1. An atomic monoid M is called a *Garside monoid* if

- (i) M is left and right cancellative;
- (ii) (M, \leq_L) and (M, \leq_R) are lattices;
- (iii) M contains an element Δ , called a *Garside element*, satisfying the following:
 - (a) for each $a \in M$, $a \leq_L \Delta$ if and only if $a \leq_R \Delta$;
 - (b) the set $\{a \in M : a \leq_L \Delta\}$ is finite and generates M .

An element a of M is called a *simple element* if $a \leq_L \Delta$. A *Garside group* is defined as the group of fractions of a Garside monoid. When M is a Garside monoid and G is the group of fractions of M , we identify the elements of M and their images in G and call them *positive elements* of G . M

is called the *positive monoid* of G , often denoted by G^+ . The triple (G, G^+, Δ) is called a *Garside structure* on G . We remark that a given group G may admit more than one Garside structure.

Let $\tau : G \rightarrow G$ be the inner automorphism of G defined by $\tau(g) = \Delta^{-1}g\Delta$ for $g \in G$. It is known that some power of τ is the identity, equivalently, some power of Δ is central.

The partial orders \leq_L and \leq_R , and thus the lattice structures in the positive monoid G^+ can be extended to the Garside group G . For $g, h \in G$, $g \leq_L h$ (resp. $g \leq_R h$) means $g^{-1}h \in G^+$ (resp. $hg^{-1} \in G^+$).

For $g \in G$, there are integers $r \leq s$ such that $\Delta^r \leq_L g \leq_L \Delta^s$. Hence, the invariants $\inf(g) = \max\{r \in \mathbb{Z} : \Delta^r \leq_L g\}$, $\sup(g) = \min\{s \in \mathbb{Z} : g \leq_L \Delta^s\}$ and $\text{len}(g) = \sup(g) - \inf(g)$ are well-defined.

For $g \in G$, we denote its conjugacy class $\{h^{-1}gh : h \in G\}$ by $[g]$. Define $\inf_s(g) = \max\{\inf(h) : h \in [g]\}$, $\sup_s(g) = \min\{\sup(h) : h \in [g]\}$ and $\text{len}_s(g) = \sup_s(g) - \inf_s(g)$.

For every element $g \in G$, the following limits are well-defined:

$$t_{\inf}(g) = \lim_{n \rightarrow \infty} \frac{\inf(g^n)}{n}; \quad t_{\sup}(g) = \lim_{n \rightarrow \infty} \frac{\sup(g^n)}{n}; \quad t_{\text{len}}(g) = \lim_{n \rightarrow \infty} \frac{\text{len}(g^n)}{n}.$$

These limits were introduced in [LL07] in studying translation numbers in Garside groups. In this paper we will exploit the following properties.

Proposition 2.2 ([LL07, LL08]). *For $g, h \in G$,*

- (i) $t_{\inf}(h^{-1}gh) = t_{\inf}(g)$ and $t_{\sup}(h^{-1}gh) = t_{\sup}(g)$;
- (ii) $t_{\inf}(g^n) = n \cdot t_{\inf}(g)$ and $t_{\sup}(g^n) = n \cdot t_{\sup}(g)$ for all integers $n \geq 1$;
- (iii) $\inf_s(g) = \lfloor t_{\inf}(g) \rfloor$ and $\sup_s(g) = \lceil t_{\sup}(g) \rceil$;
- (iv) $t_{\inf}(g)$ and $t_{\sup}(g)$ are rational of the form p/q , where p and q are coprime integers and $1 \leq q \leq \|\Delta\|$.

3. PERIODICITY AND GARSIDE STRUCTURE

For a group G , let $Z(G)$ denote the center of G . The following is the main result of this section.

Theorem 3.1. *Let G be a Garside group. Then $Z(G)$ is cyclic if and only if any pair of Garside elements of G are commensurable.*

It is easy to see that the periodicity of an element in G does not depend on the choice of a particular Garside element if and only if any pair of Garside elements of G are commensurable. Hence we have the following.

Corollary 3.2. *Let G be a Garside group. The periodicity of an element of G does not depend on the choice of a particular Garside element if and only if $Z(G)$ is cyclic.*

We prove Theorem 3.1 by using the following lemma. For $g \in G$, let $L(g) = \{a \in G^+ : a \leq_L g\}$ and $R(g) = \{a \in G^+ : a \leq_R g\}$.

Lemma 3.3. *Let (G, G^+, Δ) be a Garside structure on a group G .*

- (i) *Let c be a positive element in $Z(G)$. Then $L(c) = R(c)$.*
- (ii) *Let c be a positive element in $Z(G)$ such that $\Delta \leq_L c$. Then c is a Garside element, that is, $L(c) = R(c)$ and $L(c)$ generates the positive monoid G^+ .*

Proof. (i) Let $a \in L(c)$, then $c = ab$ for some $b \in G^+$. Because c is central, $ab = c = bcb^{-1} = b(ab)b^{-1} = ba$. Therefore $c = ba$, hence $a \in R(c)$. This means that $L(c) \subset R(c)$. Similarly, $R(c) \subset L(c)$.

(ii) By (i), $L(c) = R(c)$. As $\Delta \leq_L c$, we have $L(\Delta) \subset L(c)$. Since $L(\Delta)$ generates G^+ , so does $L(c)$. \square

Proof of Theorem 3.1. Suppose that $Z(G)$ is cyclic. Let (G, G_1^+, Δ_1) and (G, G_2^+, Δ_2) be Garside structures on G . Then there exist positive integers m_1 and m_2 such that $\Delta_1^{m_1}$ and $\Delta_2^{m_2}$ are central in G . Because $Z(G)$ is cyclic, $\Delta_1^{m_1}$ and $\Delta_2^{m_2}$ are commensurable, hence Δ_1 and Δ_2 are commensurable.

Conversely, suppose any pair of Garside elements are commensurable. Fix a Garside structure (G, G^+, Δ) on G . Let m be the smallest positive integer such that Δ^m is central.

We claim that any nonidentity element of $Z(G)$ is commensurable with Δ . Let g be a nonidentity central element. Take an integer k such that

$$k \equiv 0 \pmod{m} \quad \text{and} \quad k \geq -\inf(g) + 1.$$

Let $c = \Delta^k g$. Then c is a central element with $\Delta \leq_L c$, hence c is a Garside element by Lemma 3.3. By the hypothesis, c is commensurable with Δ . As c is central, there exist nonzero integers p and q such that $\Delta^p = c^q = (\Delta^k g)^q = \Delta^{kq} g^q$. Since $g^q = \Delta^{p-kq}$, g is commensurable with Δ .

It is known that Garside groups are torsion-free by Dehornoy [Deh98], and that every abelian subgroup of a Garside group is finitely generated by Charney, Meier and Whittlesey [CMW04]. Thus $Z(G)$ is torsion-free and finitely generated. Moreover, by the above claim, any two nonidentity elements of $Z(G)$ are commensurable because each of them is commensurable with Δ . These imply that $Z(G)$ is cyclic. \square

4. ROOTS OF PERIODIC ELEMENTS

Let G be a Garside group with Garside element Δ . First, we explore some basic properties of periodic elements regarding translation numbers.

Lemma 4.1. *Let $g \in G$ be a periodic element. Then the following hold.*

- (i) $t_{\text{len}}(g) = 0$, that is, $t_{\text{inf}}(g) = t_{\text{sup}}(g)$.
- (ii) $t_{\text{inf}}(g^k) = k \cdot t_{\text{inf}}(g)$ for all $k \in \mathbb{Z}$.
- (iii) For any $k \in \mathbb{Z}$, $\text{len}_s(g^k)$ is either 0 or 1.
- (iv) Let $t_{\text{inf}}(g) = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$ with $\gcd(p, q) = 1$. Then the following are equivalent for $k \in \mathbb{Z}$:
 - (a) g^k is conjugate to a power of Δ ;
 - (b) $\text{len}_s(g^k) = 0$;
 - (c) $t_{\text{inf}}(g^k)$ is an integer;
 - (d) k is a multiple of q .

In particular, g^q is conjugate to Δ^p .

Proof. (i) Since $g^k = \Delta^\ell$ for some $k \in \mathbb{Z}_{\geq 1}$ and $\ell \in \mathbb{Z}$,

$$t_{\text{len}}(g) = \frac{1}{k} \cdot t_{\text{len}}(g^k) = \frac{1}{k} \cdot t_{\text{len}}(\Delta^\ell) = \frac{1}{k} \cdot 0 = 0.$$

(ii) We know that $t_{\inf}(g^k) = k \cdot t_{\inf}(g)$ holds for all $k \geq 0$. Let $k < 0$, then $k = -\ell$ for some $\ell \geq 1$. Since $t_{\inf}(g) = t_{\sup}(g)$ by (i) and $t_{\inf}(h^{-1}) = -t_{\sup}(h)$ for all $h \in G$, we have

$$t_{\inf}(g^k) = t_{\inf}((g^{-1})^\ell) = \ell \cdot t_{\inf}(g^{-1}) = \ell \cdot (-t_{\sup}(g)) = k \cdot t_{\inf}(g).$$

(iii) Let $t_{\inf}(g) = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$ with $\gcd(p, q) = 1$. Choose any $k \in \mathbb{Z}$. Because $t_{\sup}(g) = t_{\inf}(g) = p/q$ by (i),

$$(2) \quad \text{len}_s(g^k) = \sup_s(g^k) - \inf_s(g^k) = \lceil t_{\sup}(g^k) \rceil - \lfloor t_{\inf}(g^k) \rfloor = \lceil kp/q \rceil - \lfloor kp/q \rfloor$$

by (ii). Therefore $\text{len}_s(g^k)$ is either 0 or 1.

(iv) It is obvious that $\text{len}_s(g^k) = 0$ if and only if g^k is conjugate to a power of Δ . By Eq. (2), $\text{len}_s(g^k) = 0$ if and only if $t_{\inf}(g^k) = kp/q$ is an integer. Because p and q are coprime, kp/q is an integer if and only if k is a multiple of q . Therefore the four conditions—(a), (b), (c) and (d)—are equivalent.

By (a) and (d), g^q is conjugate to Δ^ℓ for some integer ℓ , hence $t_{\inf}(g) = \ell/q$. Because $t_{\inf}(g) = p/q$ by the hypothesis, we have $\ell = p$. \square

Theorem 5.1 in [LL08] shows that for every $g \in G$,

$$\begin{aligned} t_{\inf}(g) &= \max\{\inf_s(g^k)/k : k = 1, \dots, \|\Delta\|\}, \\ t_{\sup}(g) &= \min\{\sup_s(g^k)/k : k = 1, \dots, \|\Delta\|\}. \end{aligned}$$

Hence the values of $t_{\inf}(g)$, $t_{\sup}(g)$ and $t_{\text{len}}(g)$ can be computed explicitly.

Remark 4.2. If g is an element of G with $t_{\text{len}}(g) = 0$, then $t_{\inf}(g) = t_{\sup}(g) = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$. Hence $\inf_s(g^q) = \sup_s(g^q) = p$, which implies that g is periodic. Therefore we can see that g is periodic if and only if $t_{\text{len}}(g) = 0$. Further, g is p/q -periodic if and only if $t_{\inf}(g) = t_{\sup}(g) = p/q$ for $p \in \mathbb{Z}$, $q \in \mathbb{Z}_{\geq 1}$ with $\gcd(p, q) = 1$.

The following is the main result of this section, which is a consequence of Lemma 4.1(iv).

Theorem 4.3. *Let G be a Garside group with Garside element Δ , and let $g \in G$ and $a, b, k \in \mathbb{Z}_{\neq 0}$.*

- (i) *If g^{kb} is conjugate to Δ^{ka} , then g^b is conjugate to Δ^a .*
- (ii) *If each of g^a and g^b is conjugate to a power of Δ , then so is $g^{\gcd(a,b)}$.*

Proof. The hypothesis in either case of (i) or (ii) implies that g is periodic. Let $t_{\inf}(g) = p/q$ for $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{\geq 1}$ with $\gcd(p, q) = 1$. Then g^q is conjugate to Δ^p by Lemma 4.1(iv).

(i) Since g^{kb} is conjugate to Δ^{ka} , one has $t_{\inf}(g) = a/b = p/q$, hence there is $d \in \mathbb{Z}_{\neq 0}$ such that $a = dp$ and $b = dq$. Therefore g^b is conjugate to Δ^a .

(ii) By Lemma 4.1(iv), both a and b are multiples of q , hence $\gcd(a, b)$ is a multiple of q . Therefore $g^{\gcd(a,b)}$ is conjugate to some power of Δ . \square

Theorem 4.3(i) implies that if $g^k = \Delta^{k\ell}$ then g is conjugate to Δ^ℓ . The following example illustrates that the general statement for the uniqueness of roots up to conjugacy (i.e. if $g^k = h^k$ for $k \neq 0$ then g is conjugate to h) does not hold in Garside groups, even for periodic elements g and h .

Example 4.4. Let G be the group defined by

$$G = \langle x, y \mid x^a = y^a \rangle, \quad a \geq 2.$$

It is a Garside group with Garside element $\Delta = x^a = y^a$ [DP99, Example 4]. (Note that x and y are periodic elements and that Δ is central.) We claim that

- (i) x and y are not conjugate;
- (ii) there is no Garside structure on G in which x is a Garside element.

Because $G/\langle\Delta\rangle = \langle x, y \mid x^a = y^a = 1 \rangle = \langle x \mid x^a = 1 \rangle * \langle y \mid y^a = 1 \rangle$, the images of x and y in $G/\langle\Delta\rangle$ are not conjugate. Therefore x and y are not conjugate.

Assume that there exists a Garside structure on G with Garside element x . Because $x^a = y^a$ and x is a Garside element, y is conjugate to x by Theorem 4.3(i). It is a contradiction to (i).

The above example shows that

there is a Garside group with a periodic element g such that there is no Garside structure in which g is a Garside element.

Therefore it gives a negative answer to the question of Bessis stated in §1.

5. FINITE SUBGROUPS OF THE QUOTIENT GROUP G_Δ

Let G be a Garside group with Garside element Δ , and let m be the smallest positive integer such that Δ^m is central in G . Let G_Δ denote the quotient $G/\langle\Delta^m\rangle$. For an element $g \in G$, let \bar{g} denote the image of g under the natural projection from G to G_Δ . The following theorem was proved by Bestvina [Bes99, Theorem 4.5] for Artin groups of finite type, and then proved by Charney, Meier and Whittlesey [CMW04, Corollary 6.8] for Garside groups.

Theorem 5.1 ([Bes99, CMW04]). *The finite subgroups of G_Δ are, up to conjugacy, one of the following two types:*

- (i) *the cyclic group generated by the image of $\Delta^u a$ in G_Δ for some $u \in \mathbb{Z}$ and some simple element $a \neq \Delta$ such that if $a \neq 1$, then for some integer $2 \leq q \leq \|\Delta\|$*

$$\tau^{(q-1)u}(a) \tau^{(q-2)u}(a) \cdots \tau^u(a) a = \Delta;$$

- (ii) *the direct product of a cyclic group of type (i) and $\langle \bar{\Delta}^k \rangle$ where Δ^k commutes with a .*

In the case of Artin groups of finite type, Bestvina showed that finite subgroups of G_Δ are all cyclic groups (hence they are of type (i) in the above theorem). Using the following lemma, we show in Theorem 5.3 that the same is true for Garside groups.

Lemma 5.2. *Let H be an abelian subgroup of G which consists of periodic elements. Then $t_{\inf}|_H : H \rightarrow \mathbb{Q}$ is a monomorphism. In particular, H is a cyclic group.*

Proof. Let $h_1, h_2 \in H$ with $t_{\inf}(h_i) = p_i/q_i$ for $i = 1, 2$. Because $h_i^{q_i}$ is conjugate to Δ^{p_i} (by Lemma 4.1) and Δ^m is central, one has $h_i^{q_i m} = \Delta^{p_i m}$ for $i = 1, 2$. Therefore

$$(h_1 h_2)^{q_1 q_2 m} = (h_1^{q_1 m})^{q_2} \cdot (h_2^{q_2 m})^{q_1} = \Delta^{p_1 m q_2} \Delta^{p_2 m q_1} = \Delta^{m(p_1 q_2 + p_2 q_1)},$$

hence $t_{\inf}(h_1 h_2) = m(p_1 q_2 + p_2 q_1)/q_1 q_2 m = p_1/q_1 + p_2/q_2 = t_{\inf}(h_1) + t_{\inf}(h_2)$. This means that $t_{\inf}|_H : H \rightarrow \mathbb{Q}$ is a homomorphism. If $h \in H$ and $t_{\inf}(h) = 0$, then h is conjugate to $\Delta^0 = 1$ by Lemma 4.1, hence $h = 1$. This means that $t_{\inf}|_H : H \rightarrow \mathbb{Q}$ is injective.

Notice that, for all $g \in G$, $t_{\inf}(g)$ is rational of the form p/q with $1 \leq q \leq \|\Delta\|$ (see Proposition 2.2). Therefore $t_{\inf}(H)$ is a discrete subgroup of \mathbb{Q} , hence it is a cyclic group. Because $t_{\inf}|_H : H \rightarrow \mathbb{Q}$ is injective, H is also a cyclic group. \square

Theorem 5.3. *Let G be a Garside group with Garside element Δ . Then every finite subgroup of G_Δ is cyclic.*

Proof. Let K be a finite subgroup of G_Δ . Let H be the preimage of K under the natural projection $G \rightarrow G_\Delta$. Notice that, by Theorem 5.1, H is abelian and every element of H is periodic. By Lemma 5.2, H is a cyclic group, hence K is cyclic. \square

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